Optimal Utilization Of Market Forecasts And The Evaluation Of Investment Performance

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Abstract

The purpose of this paper is to examine the problem faced by the portfolio manager attempting to optimally incorporate forecasts of future market returns into his portfolio. Given the solution to this problem we then shall focus our attention on the problem involved in measuring a portfolio manager's ability when he is explicitly engaged in forecasting the prices on individual securities (i.e., security analysis) and in forecasting the future course of market prices (i.e., "timing activities"). We shall consider these problems here in the context of the Sharpe-Lintner mean-variance general equilibrium model of the pricing of capital assets, and in the context of the expanded two factor version of the Sharpe model suggested by Black, Jensen and Scholes (1972) In addition we shall concentrate our attention here on an investigation of just what can and cannot be said about portfolio performance solely on the basis of data on the time series of portfolio and market returns.

In section 2 we outline the foundations of the analysis and its relationship to the general equilibrium structure of security prices given by the Sharpe-Lintner model. In section 3 we briefly summarize the measure of security selection ability suggested by Jensen (1968). Section 4 contains a solution to the problem of the optimal incorporation of market forecasts into portfolio policy and provides the structure for the analysis in section 5 of the measurement problems introduced into the evaluation of portfolio performance by market forecasting activities by the portfolio manager. Section 6 presents the complete development of the model within the two factor equilibrium model of the pricing of capital assets suggested by Black (1970) and Black, Jensen and Scholes (1972). Section 7 contains a brief summary of the conclusions of the analysis.

Mathematical Methods in Finance, G. P. Szego and Karl Shell, Editors
(North-Holland Publishing Company, 1972).

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1. Introduction

A number of authors in the recent past[1] have considered the problem of evaluating the performance of the managers of investment portfolios. The purpose of this paper is to examine the problem faced by the portfolio manager attempting to optimally incorporate forecasts of future market returns into his portfolio. Given the solution to this problem we then shall focus our attention on the problem involved in measuring a portfolio manager's ability when he is explicitly engaged in forecasting the prices on individual securities (i.e., security analysis) and in forecasting the future course of market prices (i.e., 'timing activities'). We shall consider these problems here in the context of the Sharpe (1964) — Lintner (1965) mean-variance general equilibrium model of the pricing of capital assets, and in the context of the expanded two factor version of the Sharpe model suggested by Black, Jensen and Scholes (1972) In addition we shall concentrate our attention here on

[1] Cf. Treynor (1965), Treynor and Mazuy (1966), Sharpe (1966), Jensen (1968; 1969), and Friend and Blume (1970).

* Associate Professor, Graduate School of Management, University of Rochester, Rochester, New York, U.S.A. I am indebted to Fischer Black, Jack Treynor, John Long, and the members of the Finance Workshop at the University of Rochester for helpful comments and to the Ford and National Science Foundations for financial support.

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2. Foundations Of The Model[2]

Let

$\tilde{R}_{jt} = \tilde{r}_{jt} - r_{Ft} =$ excess return on the jth asset in time t; $\tilde{r}_{jt}$ = the total returns (dividends plus capital gains) on the jth asset in time t, and r_{Ft} = the riskless rate of interest for time t. (1a)

$\tilde{R}_{Mt} = \tilde{r}_{Mt} - r_{Ft} =$ excess returns on the market portfolio in time t. (1b)

[2] Sections 2-5 are a reformulation and extension of some of the material which appears in Jensen (1968, pp. 395-96). I am indebted to John Lintner for pointing out the existence of an error there. The reformulation which appears here corrects that error and makes clear some assumptions which were only implicit in the previous paper.

$E(\tilde{R}_M) = E(\tilde{r}_{Mt}) - r_{Ft}$: that is we assume the expected excess return on the market portfolio is constant through time. (2)

$\tilde{R}_{jt} = E(\tilde{R}_{jt}) + b_j\tilde{\pi}_t + \tilde{e}_{jt}$: the 'market model' formulation, (3)

where b_j is a parameter which may vary from security to security and $\tilde{\pi}_t$ is an unobservable 'market factor' which to some extent affects the returns on all securities. We also assume

$$E(\tilde{\pi}_t) = 0 \tag{4a}$$

$$E(\tilde{e}_{jt}) = 0 \qquad j = 1,2,...,N \tag{4b}$$

$$\text{cov}(\tilde{\pi}_t, \tilde{e}_{jt}) = 0 \qquad j = 1,2,...,N \tag{4c}$$

$$\text{cov}(\tilde{e}_{jt}, \tilde{e}_{it}) = \begin{cases} 0 & j \neq i \\ \sigma^2(\tilde{e}_j) & j = i \end{cases} \qquad j = 1,2,...,N \tag{4d}$$

Now using arguments similar to those of Jensen (1969) it can be shown that to a close approximation the excess returns on the market portfolio can be expressed as:

$$\tilde{R}_{Mt} \cong E(\tilde{R}_M) + \tilde{\pi}_t \tag{5}$$

The Sharpe (1964), Lintner (1965), Mossin (1966) models of the pricing of capital assets under uncertainty imply that

$$E(\tilde{R}_{jt}) = \beta_j E(\tilde{R}_M), \tag{6}$$

where

$$\beta_j \frac{\text{cov}(\tilde{R}_{jt}, \tilde{R}_{Mt})}{\sigma^2(\tilde{R}_{Mt})}$$

and as shown by Fama (1968) and Jensen (1969) $\beta_j \cong b_j$. Using this result and substituting from (5) into (6) for $E(\tilde{R}_M)$ and adding $\beta_j \tilde{\pi}_t + \tilde{e}_{jt}$ to both sides of (6), we have

$$E(\tilde{R}_{jt}) + \beta_j \tilde{\pi}_t + \tilde{e}_{jt} \cong \beta_j [\tilde{R}_{Mt} - \tilde{\pi}_t] + \beta_j \tilde{\pi}_t + \tilde{e}_{jt} \qquad (7)$$

Using (3) and the fact that $\beta_j \cong b_j$ the LHS of (7) is just $\tilde{R}_{jt}$. Hence (7) reduces to

$$\tilde{R}_{jt} = \beta_j \tilde{R}_{Mt} + \tilde{e}_{jt} \qquad (8)$$

Thus, if the asset pricing model given by (6) and the market model given by (3) are valid, (8) says that the realized excess returns on any security or port- folio can be expressed as a linear function of its systematic risk, the realized excess returns on the market portfolio and a random error, $\tilde{e}_{jt}$, which has an expected value of zero.

3. A Measure Of A Portfolio Manager's Ability To Forecast The Prices Of Individual Securities

Eq. (8) can be estimated for a managed portfolio. However, if the manager is able to forecast individual security prices he will tend to systematically select securities which realize $\tilde{e}_{jt} > 0$. Hence his portfolio will earn more than the 'normal' risk premium for its level of risk. We allow for this possibility by not constraining the estimating regression to pass through the origin. That is, we allow for the existence of a non-zero constant in (8) by using (9) as the estimating equation:

$$\tilde{R}_{jt} = \alpha_j + \beta_j \tilde{R}_{Mt} + \tilde{u}_{jt} \qquad (9)$$

where $\tilde{u}_{jt}$ is assumed to have zero expectation and to be independent of $\tilde{R}_{Mt}$. The intercept α_j measures the increment in average returns due to the manager's security selection abilities (assuming for now that he does not attempt to forecast general movements in market prices). As such it represents a measure of the manager's ability to forecast individual security prices (cf. Jensen (1968) for a detailed discussion of these issues).

4. Optimal Portfolio Adjustment To Market Forecasts

Assume for the moment that all investors with an interest in a given portfolio have identical preference functions[3], $U\left[E(\tilde{R}_j),V(\tilde{R}_j)\right]$, involving only the single period mean, $E(\tilde{R}_j)$, and variance, $V(\tilde{R}_j)$, of portfolio excess return. The portfolio manager should incorporate his forecasts, $\tilde{\pi}^*$, of the market factor, $\tilde{\pi}_t$, into his portfolio in a manner which will result in maximum expected utility for the portfolio shareholders. The forecast, π^*, of the market value of the market factor $\tilde{\pi}_t$ for period t based on the information set available to the manager $\phi_{j,t-1}$ at time $t-1$.

$$\tilde{\pi}_t^* = E(\tilde{\pi}_t | \phi_{j,t-1}) \tag{10}$$

Thus, the market excess return expected by the manager of portfolio j is $E_j(\tilde{R}_{Mt}) = (\tilde{R}_M) + \tilde{\pi}_t^*$. We shall represent the variance of this conditional distribution by $\sigma_j^2(\tilde{\pi}_t)$, where

$$\sigma_j^2(\tilde{\pi}_t) = Var(\tilde{\pi}_t | \phi_{j,t-1}) \tag{11}$$

Assuming that the manager has no special information about the future returns on individual securities his problem is to decide upon the division of the portfolio's assets between the market portfolio and the riskless asset. Let γ_t be the fraction invested in the

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[3] In reality this assumption is not as restrictive as it seems since many portfolios have only one investor and when we consider mutual funds it can still be approximately valid if the funds announce investment policies in advance and then investors distribute them- selves across funds according to the matching of preferences and policies.

market portfolio at time t and $1-\gamma$ the fraction invested in the riskless asset. Under the assumptions that there are no transactions costs, and no restrictions on borrowing, lending, and short selling the expected excess return and variance of return on the portfolio are

$$E(\tilde{R}_{jt}) = \gamma_t [E(\tilde{R}_M) + \tilde{\pi}_t^*] \tag{12a}$$

$$V(\tilde{R}_{jt}) = \gamma_t^2 \sigma_t^2(\tilde{\pi}_t) \tag{12b}$$

The manager's problem is to

$$\max_{\gamma_t} U[E(\tilde{R}_{jt}), V(\tilde{R}_{jt})] = \max_{\gamma_t} U[\gamma_t(E(\tilde{R}_M) + \tilde{\pi}_t^*), \gamma_t^2 \sigma_t^2(\tilde{\pi}_t^*)] \tag{13}$$

and the solution to this yields

$$\gamma_t = \frac{1}{2\sigma_j^2(\tilde{\pi}_t)} \frac{dV(R_j)}{dE(R_j)} [E(\tilde{R}_M) + \tilde{\pi}_t^*] \tag{14}$$

where

$\dfrac{dV(R_j)}{dE(R_j)} = -\dfrac{\partial U}{\partial E(R_j)} \Big/ \dfrac{\partial U}{\partial V(R_j)}$ > 0 is minus the marginal rate of substitution of variance for expected excess return for each of the portfolio's investors (or the slope of the indifference curve between variance and excess return).

Now since $\beta_M = 1$ and $\beta_{jt} = \gamma_t \beta_M$ we see that $\beta_{jt} = \gamma_t$. Thus, at each point in time the managers optimal choice of systematic risk for the portfolio, β_{jt}, is given by

$$\beta_{jt} = \theta_{jt} E(\tilde{R}_M) + \theta_{jt} \tilde{\pi}_t^*, \tag{15}$$

where

$$\theta_{jt} = \frac{1}{2\sigma_j^2(\tilde{\pi}_t)} \frac{dV(R_j)}{dE(R_j)} .$$

Thus, the size of θ_{jt} determines the degree to which he allows his forecasts to affect the portfolio risk level, and it in turn depends on his uncertainty regarding his forecast, $\sigma_j^2(\tilde{\pi}_t)$ and his investor's willingness to bet on his forecasts as summarized by $dV(R_j)/dE(R_j)$ (the slope of their indifference curves between mean and variance). The larger is the uncertainty of his forecast as measured by $\sigma_j^2(\tilde{\pi}_t)$, the smaller will be the amount by which he changes the risk level to incorporate his forecasts. In addition the more averse to marginal increments of risk are his investors the lower is $dV(R_j)/dE(R_j)$ and the smaller will be the amount by which he changes the risk level.

Let us assume that the manager's forecasts and the market factor follow a bivariate normal distribution. Then, in general

$$\tilde{\pi}_t = d_0 + d_1\tilde{\pi}_t^* + \tilde{v}_{jt}, \tag{16}$$

where d_0 and d_1 are constants which correct for any systematic biases in the manager's forecasts, $\tilde{v}_{jt}$ is normal with $E(\tilde{v}_{jt}) = 0$ and we assume $\text{cov}(\pi_t^*, \tilde{v}_{jt}) = 0$. If d_0 and d_1 are zero and unity respectively the manager's forecasting technique tends neither to produce estimates which are systematically high or low $(d_0 = 0)$, nor does he systematically under- or over-estimate the magnitude of any market movement $(d_1 = 1)$. The manager could estimate d_0 and d_1 by regression techniques given a file of past forecasts and if $d_0 \neq 0$ or $d_1 \neq 1.0$ the optimal utilization of the forecasts requires adjustment of the forecasts π_t^*. Let $\tilde{\pi}_t^{'}$ be the adjusted forecast defined by

$$\begin{aligned} \pi_t^{'} &= d_0 + d_1\pi_t^* \\ &= \tilde{\pi}_t - \tilde{v}_{jt} \end{aligned} \tag{17}$$

where we assume here that the manager knows the true coefficients d_0 and d_1. Substituting $\pi_t^{'}$ given by (17) for π_t^* in (15) we have

$$\beta_{jt} = \theta_{jt} E(\tilde{R}_M) + \theta_{jt}\pi'_t \tag{18}$$

and now using (16) and (17) we see that $\sigma_j^2(\tilde{\pi}_t) = \sigma_j^2(\tilde{v}_{jt})$; and if we assume that $\sigma^2(\tilde{v}_{jt}) = \sigma^2(\tilde{v}_j)$ for all t, θ_{jt} is given by

$$\theta_{jt} = \frac{1}{2\sigma^2(\tilde{v}_j)} \frac{dV(R_j)}{dE(R_j)}, \tag{19}$$

and if $dV(R_j)/dE(R_j)$ is a constant independent of $V(R_j)$ or $E(R_j)$ (or approximately so)[4], we can write (18) as

$$\beta_{jt} = \beta_j + 0_j \tilde{\pi}'_t, \tag{20}$$

and $\beta_j = 0_j E(R_M)$ can thus be considered his 'target' risk level—that risk level which on the average the manager wishes to maintain given the unconditional expected returns on the market portfolio and the preferences of his shareholders.

If the manager in addition to forecasting market returns, also believes he has information about individual securities he will not simply lever the market portfolio up or down. He can construct a portfolio he views as optimal given his information set, call it *P*, and this will not be the market portfolio. He might then Speculate On his market forecasts by levering the portfolio *P* up or down.[5] The analysis is analogous to that performed above. However, since β_P need not equal unity, we find that

$$\beta_{jt} = \gamma_t \beta_{Pt}$$

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[4] $dV(R_j)/dE(R_j)$ is, of course, literally constant for the special case of constant absolute risk aversion on the part of investors. For other utility functions, to the extent that we ignore changes in the value of $dV(R_j)/dE(R_j)$ associated with changes in $E(R_j)$ and $V(R_j)$ we are ignoring only the second order effects, so for small changes in $E(R_j)$ and $V(R_j)$ this approximation is probably not bad.

[5] However, this procedure is not likely to be an optimal one (cf. F. Black and J. Treynor, 'How to use security analysis to improve portfolio selection', in proceedings: *Seminar on the analysis of security prices*,

and again assuming that $\sigma^2(\tilde{v}_{jt}), dV(R_j)/dE(R_j),$ and β_{Pt} are constant through time we get all expression for β_{jt}, which is identical to (20) except that

$$0_j = \frac{\beta_P}{2\sigma^2(\tilde{v}_j)} \frac{dV(R_j)}{dE(R_j)}. \qquad (21)$$

Thus, if the manager levers the portfolio *P* instead of the market portfolio there is little difference in the analysis. Henceforth we shall assume that the manager is engaged in both security analysis and market forecasting activities and we leave the definition of 0_j to be whichever is relevant for a given situation.

5. Effects Of A Manager's Ability To Predict The Market Factor π

5. 1. Effects on portfolio returns

Utilizing the results obtained above we find that the process generating the returns on a portfolio whose manager is engaged in both security analysis and market forecasting activities can be represented by

$$\tilde{R}_{jt} = \alpha_j + \tilde{\beta}_{jt}\tilde{R}_{Mt} + \tilde{u}_{jt} = \alpha_j + \left(\beta_j + \theta_j \pi_t'\right)\tilde{R}_{Mt} + u_{jt}. \qquad (22)$$

If the manager can forecast the market factor ft and acts upon his forecast in a rational manner, he will, of course, be able to increase the returns on his portfolio. To see the effects of his forecasting ability on the expected returns of the portfolio we take the expected value of (22) to obtain

$$E\left(\tilde{R}_{jt}\right) = \alpha_j + \beta_j E\left(\tilde{R}_M\right) + \theta_j E\left[\tilde{\pi}_t'\left(E\left(\tilde{R}_M\right) + \tilde{\pi}_t\right)\right]$$

$$= \alpha_j + \beta_j E\left(\tilde{R}_M\right) + \theta_j \rho^2 \sigma^2(\tilde{\pi}), \qquad (23)$$

University of Chicago Graduate School of Business, November, 1967), but in order to obtain tractable results we shall assume that this procedure is approximately descriptive of the manager's policies.

where ρ is the correlation between the manager's unadjusted forecasts $\tilde{\pi}_t^*$ and $\tilde{\pi}_t$. Eq. (23) follows from the fact that $E(\tilde{\pi}_t') = 0$ and

$$E(\tilde{\pi}_t'\tilde{\pi}_t) = \text{cov}(\tilde{\pi}', \tilde{\pi})$$

$$= d_1 \text{cov}(\tilde{\pi}^*, \tilde{\pi})$$

$$= \rho^2 \sigma^2(\tilde{\pi})$$

$$\text{cov}(\tilde{\pi}^*, \tilde{\pi}) = \rho\sigma(\tilde{\pi}^*)\sigma(\tilde{\pi})$$

$$d_1 = \rho\sigma(\tilde{\pi}) / \sigma(\tilde{\pi}^*).$$

Now by (6) we know that $\beta_j E(R_M)$ represents the expected compensation for the average risk level of the portfolio. Furthermore, as we saw above, α_j represents the amount by which the portfolio returns are increased as a result of the manager's ability to select undervalued securities. Thus, the last term in (23), $\theta_j \rho^2 \sigma^2(\tilde{\pi})$, represents the expected increment in the portfolio returns which is due solely to the manager's ability to forecast the unexpected market returns, $\tilde{\pi}$. Given that the manager can forecast to some extent, his profit opportunities are proportional to $\sigma^2(\tilde{\pi})$, the variance of the market factor. In addition to $\sigma^2(\tilde{\pi})$ the profits depend on θ_j (which involves the residual uncertainty $\sigma^2(\tilde{v}_j)$ and the shareholders risk-return trade off $dV(R_j) / dE(R_j)$ and ρ, the correlation between his forecasts and the actual market returns. Since $\sigma^2(\pi)$ and θ_j are given exogenously to the manager, we see that the profits from his forecasting ability are directly related to p, and thus his forecasting ability is totally summarized by ρ.[6]

6 Black and Treynor [·] have also obtained this result.

5.2. Effects on the estimated risk coefficient

The large sample least squares estimate of β_j for a time series of returns, $t = 1,2,\ldots,T$ is

$$P\lim \hat{\beta}_j = P\lim \frac{\hat{\text{cov}}\left[\tilde{R}_{jt}, \tilde{R}_{Mt}\right]}{\hat{\sigma}^2\left(\tilde{R}_{Mt}\right)}$$

$$= P\lim \frac{\sum_{t=1}^{T}\left[\tilde{R}_{jt}\left(\tilde{R}_{Mt} - \overline{R}_M\right)\right]}{\sum_{t=1}^{T}\left[\tilde{R}_{Mt} - \overline{R}_M\right]^2}$$

$$= P\lim \frac{\sum_{t=1}^{T}\left[\left\{\alpha_j + \beta_j\left(E\left(\tilde{R}_M\right) + \tilde{\pi}_t\right) + \theta_j \tilde{\pi}_t^{'}\left(E\left(\tilde{R}_M\right) + \tilde{\pi}_t\right) + \tilde{u}_{jt}\right\}\tilde{R}_{Mt} - \overline{R}_M\right]}{\sum_{t=1}^{T}\left[\tilde{R}_{Mt} - \overline{R}_M\right]^2}$$

$$= \beta_j + \theta_j\left[\frac{\text{cov}\left(\tilde{\pi}^{'}, \tilde{\pi}\right)E\left(\tilde{R}_M\right) + E\left(\tilde{\pi}^{'}, \tilde{\pi}^2\right)}{\sigma^2\left(\tilde{\pi}\right)}\right]$$

$$= \beta_j + \theta_j\left[\rho^2 E\left(\tilde{R}_M\right)\frac{E\left(\tilde{\pi}^{'3}\right)}{\sigma^2\left(\tilde{\pi}\right)}\right] \qquad (24)$$

since $P\lim \tilde{R}_{Mt} - \overline{R}_M = E\left(R_M\right) + \tilde{\pi}_t - E\left(R_M\right) = \tilde{\pi}_t$. Thus, the estimate of β_j will be unbiased only if the manager cannot forecast market movements. If he has no forecasting ability ρ will be zero, and in addition $E\left(\tilde{\pi}^{'3}\right)$ will also be zero since by eq. (17) $\tilde{\pi}^{'}$ will always be equal to $E\left(\tilde{\pi}\right) = 0$, a constant when $d_1 = 0$. Since we have assumed he optimally adjusts his forecasts by d_0 and d_1, this is not surprising since he is assumed then to know he cannot forecast, and in fact under these conditions $\beta_{jt} = \beta_j$ is a constant for all *t*. Somewhat more surprising is the fact that if we assume the manager does not go through the forecast adjustment process discussed in section 4, but instead acts upon his forecasts according to (15) even when $\rho = 0$ this same result holds.

If the manager simply uses his unadjusted forecast π_t^* then by the bivariate normality of π_t^* and π_t, we can use the other regression

$$\tilde{\pi}_t^* = d_0 + d_1\tilde{\pi}_t + \tilde{v}'_{jt} \tag{25}$$

to obtain the portfolio return generating process in (25) d_0' and d_1' are regression coefficients, and $\tilde{v}'_{jt}$ is a normally distributed random error with $E(\tilde{v}_{jt}) = 0$ and $\operatorname{cov}(\tilde{v}_{jt}, \tilde{\pi}_t) = 0$. Again, if d_0' and d_1', are zero and unity respectively the estimates are unbiased. Substituting for $\tilde{\pi}_t^*$ from (25) into (15) (and again assuming that θ_{jt} is constant for all $t, V(R_j)$ and $E(R_j)$) we find the riskiness of the portfolio to be given by

$$\tilde{\beta}_{jt} = \beta_j + a_j\tilde{\pi}_t + \tilde{w}_{jt}, \tag{26}$$

where $a_j = \theta_j d_1, \tilde{w}_{jt} = \theta_j \tilde{v}'_{jt}$ and $\beta_j = \theta_j\left(E\left(\tilde{R}_M\right) + d_0\right)$ can now be considered his 'target' risk level. Under these conditions the portfolio return generating process is given by

$$\tilde{R}_{jt} = \alpha_j + \left(\beta_j + a_j\tilde{\pi}_t + \tilde{w}_{jt}\right)\tilde{R}_{Mt} + \tilde{u}_{jt} \tag{27}$$

and since $\tilde{v}'_{jt}$ in (25) is independent of $\tilde{\pi}_t$ the error $\tilde{w}_{jt}$ in (27) is uncorrelated With $\tilde{\pi}_t$ (which is certainly reasonable since if it were correlated, the forecast could be improved). The large sample least squares estimate of β_j for a time series of returns, $t = 1, 2, \ldots, T$ is

$$\operatorname*{P\lim}_{T\to\infty} \hat{\beta}_j$$

$$= \operatorname*{P\lim}_{T\to\infty} \hat{\beta}_j \frac{\sum_t \left[\left\{\alpha_j + \beta_j\left(E\left(R_M\right) + \tilde{\pi}_t\right) + \left(a_j\tilde{\pi}_t + \tilde{w}_{jt}\right)\left(E\left(R_M\right) + \tilde{\pi}_t\right) + \tilde{u}_{jt}\right\}\left\{\tilde{R}_{Mt} - \overline{R}_M\right\}\right]}{\sum_t \left[\tilde{R}_{Mt} - \overline{R}_M\right]^2}$$

$$= \beta_j + a_j\left[E\left(R_M\right)\frac{E\left(\tilde{\pi}^3\right)}{\sigma^2\left(\tilde{\pi}\right)}\right], \tag{28}$$

Thus, we see that if the manager acts on his estimates even when he cannot forecast (i.e., $d_1^{'} = 0$ or $\rho = 0$) we see that our estimate of β_j is unbiased since $a_j = \theta_j d_1$ will be zero in this case.

Note, however, that if the manager can forecast the market factor to some extent that $\hat{\beta}_j$ will be a positively biased estimate of β_j under either set of assumptions if ρ and $d_1^{'}$ are positive since $E(\tilde{R}_M) > 0$ always, and if $E(\tilde{\pi}'^3)$ and $E(\tilde{\pi}^3)$ are not zero they are most likely to be positive due to the lower bound of—100% on the returns. These biases are serious since they will affect out estimates of α_j. We shall consider this issue below. For complete- ness we note here that the mean portfolio returns for the case in which the manager simply uses his unadjusted raw forecasts is given by the expected value of (27)

$$E(\tilde{R}_{jt}) = \alpha_j + \beta_j E(\tilde{R}_M) + a_j \sigma^2(\tilde{\pi}). \tag{29}$$

Again the last term, $a_j \sigma^2(\tilde{\pi})$, represents the expected increment in returns which is due solely to the manager's ability to forecast the unexpected market returns.

5.3. Effects on the measurement of the manager's stock selection ability

The large sample estimate of α_j obtained from applying traditional least squares procedures to (22) (which assumes optimal forecast adjustment) is

$$P\lim \hat{\alpha} = E(R_j) - \hat{\beta}_j E(R_M) = \alpha_j + \theta_j \rho^2 \sigma^2(\tilde{\pi}) - \theta_j \left[\rho^2 E(\tilde{R}_M)^2 + \frac{E(\tilde{\pi}'^3)}{\sigma^2(\pi)} E(\tilde{R}_M) \right], \tag{30}$$

by substitution from (23) and (24). Now, by our previous arguments, if the manager cannot successfully forecast future market returns, the last two terms on the RHS of (30) will be zero and hence our estimate of the increment in portfolio returns due to the manager's stock selection ability, α_j, will be unbiased. This also holds true in the non-forecast adjustment case (which henceforth we shall refer to as the naive forecast model) where

$$P\lim \hat{\alpha} = E(R_j) - \hat{\beta}_j E(\tilde{R}_M) = \alpha_j + a_j\sigma^2(\tilde{\pi}) - a_j\left[E(\tilde{R}_M)^2 + \frac{E(\tilde{\pi}^3)}{\sigma^2(\pi)}E(\tilde{R}_M)\right], \qquad (31)$$

is the estimate obtained from applying least squares to (27) (by substitution from (28) and (29)). In this case if the manager cannot forecast a_j will equal 0 and $\hat{\alpha}$ is therefore unbiased for this case as well.

However, if in either case the manager can forecast future market returns to sonic extent the simple time series regression technique will not allow us to separate the incremental returns due to his stock selection ability from the incremental returns due to his ability to forecast the market (the first two terms in (30) or (31)). This follows from the fact that ρ^2 in (30) and a_j in (31) will both be non-zero (and for the relevant case a_j in (31) will be positive). Furthermore, we will in general not even be able to obtain an unbiased estimate of the sum of the two components if the manager can successfully forecast the market. By using short time intervals for our time series observations (say weekly or monthly data), or by using continuously compounded rates, we can probably eliminate the negative bias due to the term involving $E(\tilde{\pi}'^3)$ in (30) or $E(\tilde{\pi}^3)$ in (31), since either procedure will tend to yield symmetric return distributions for which $E(\tilde{\pi}'^3)$ and $E(\tilde{\pi}^3)$ will be zero. However, since $E(R_M)^2$ will always be positive, we will still underestimate the total increment in returns from the managers ability by an amount equal to $\theta_j \rho^2 E(\tilde{R}_M)^2$ for the optimal forecast adjustment case and by $a_j E(\tilde{R}_M)^2$ for the naive forecast case.

It is worthwhile to reiterate the point that these bias problems arise only when we are considering a manager who can actually forecast the market. The mere fact that a manager attempts to forecast future market returns and shifts his portfolio's risk level in an attempt to capitalize on these forecasts does not hinder the measurement of Ns stock selection ability if his forecasts are in fact worthless.

5.4. An 'unbiased' measure of stock selection ability[7]

If the manager's forecasting and decision periods are coincident with the periods over which we measure his portfolio returns and if the manager follows the naive forecasting model we can separate and obtain unbiased measures of his stock selection and market forecasting abilities. Rewriting (27) in terms of $\tilde{\pi}$ we have

$$\tilde{R}_{jt} = \alpha_j + \left(\beta_j + \tilde{w}_{jt}\right)E\left(\tilde{R}_M\right) + \left[\beta_j + \tilde{w}_{jt} + a_j E\left(\tilde{R}_M\right)\right]\tilde{\pi}_t + a_j \tilde{\pi}_t^2 + \tilde{u}_{jt}. \quad (32)$$

Assuming we know $E\left(\tilde{R}_M\right)$ and can thus measure $\tilde{\pi}_t$, we can run the quadratic regression[8]

$$\tilde{R}_{jt} = \eta_0 + \eta_1 \tilde{\pi}_t + \eta_2 \tilde{\pi}_t^2 + \tilde{v}_{jt}. \quad (33)$$

if $\tilde{\pi}_t$ is symmetrically distributed about zero $\tilde{\pi}_t$ and $\tilde{\pi}_t^2$ are uncorrelated and the large sample estimates of the coefficients in (33) are

$$P\lim \hat{\eta}_0 = \alpha_j + \beta_j E\left(\tilde{R}_M\right) \quad (34a)$$

$$P\lim \hat{\eta}_1 = \beta_j + a_j E\left(\tilde{R}_M\right) \quad (34b)$$

$$P\lim \hat{\eta}_2 = a_j \quad (34c)$$

[7] The essence of the approach suggested here is similar to that first suggested by Treynor and Mazuy (1966).

Given these estimates our estimate of the manager's contribution to the portfolio return through his stock selection ability is

$$\alpha_j = \hat{\eta}_0 - \left[\hat{\eta}_1 - \hat{\eta}_2 \overline{R}_M\right] \overline{R}_M . \tag{35}$$

From (14) we can see that the estimate of the manager's contribution to the portfolio returns through his market forecasting activities is given by

$$\hat{\eta}_2 \hat{\sigma}^2 (\tilde{\pi}) \tag{36}$$

However, if the manager follows the optimal forecast adjustment model estimation of the quadratic equation (33) will not enable us to separate the returns due to market forecasting from the returns due to security analysis or to obtain an unbiased measure of the sum of the two components. Rewriting (22) in terms of $\tilde{\pi}_t$ and $\tilde{\pi}_t^2$ we obtain

$$\tilde{R}_{jt} = \alpha_j + \beta_j E(\tilde{R}_M) + \left(\beta_j + \theta_j E(\tilde{R}_M) - \theta_j \tilde{v}_j\right)\tilde{\pi}_t + \theta_j \tilde{\pi}_t^2 - \theta_j E(\tilde{R}_M)\tilde{v}_j + \tilde{u}_{jt} . \tag{37}$$

Now, if we run the quadratic regression given by eq. (33) (assuming still that $\tilde{\pi}$ and $\tilde{v}$ are symmetrically distributed) the large sample coefficient estimates are:

$$P\lim \hat{\eta}_0 = \alpha_j + \beta_j E(\tilde{R}_M) + \theta_j (\rho^2 - 1)\sigma^2(\tilde{\pi}) \tag{38a}$$

$$P\lim \hat{\eta}_1 = \rho^2 \theta_j E(\tilde{R}_M) + \beta_j \tag{38b}$$

$$P\lim \hat{\eta}_2 = \theta_j \tag{38c}$$

and since we now have four unknowns and only three equations we cannot solve for the parameters of interest. In this situation we need to have exogenous knowledge of ρ in order to provide a complete breakdown of the manager's performance. This, of course,

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[8] For large samples, of course, a_j and hence as long as the distributions are stationary there are few problems with measuring a_j .

will require more data than just the time series of portfolio and market returns. If we had data on the time series of the manager's forecasts, π_t^*, then we could estimate ρ directly and we could then solve the performance measurement problem in a fairly straightforward way. However, in general, this type of information will probably be extremely difficult if not impossible to obtain.

5.5. A temporal aggregation problem

In the previous analysis we have implicitly assumed that the manager forecasts future market movements over the next unit time interval and then suitably adjusts his portfolio—all at the beginning of the time interval. More importantly we also implicitly assumed that the time interval used by the manager in these activities is identical to the observation interval from which our return data is obtained. Of course it is unlikely that these conditions will ever be met exactly, and the question arises as to what difficulties this introduces into the performance estimation procedure. Let us assume that the manager forecasts market changes over each period (and thus eq. (32) applies to single period intervals), but we measure returns over an interval of n periods. Adding a subscript τ to refer to the period within the tth observation interval and summing[9] over τ to obtain $\tilde{R}_{jt}$ for the naïve forecast model we have

$$\tilde{R}_{jt} = \sum_{\tau=1}^{n} \tilde{R}_{jt\tau}$$

$$= n\alpha_j + \sum_{\tau}\left(\beta_j + \tilde{w}_{jt\tau}\right)E\left(\tilde{R}_{Mt\tau}\right) + \sum_{\tau}\left[\beta_j + \tilde{w}_{jt\tau} + a_j E\left(\tilde{R}_{Mt\tau}\right)\right]\tilde{\pi}_{t\tau}$$

$$+ \sum_{\tau} a_j \tilde{\pi}_{t\tau}^2 + \sum_{\tau} \tilde{u}_{jt\tau}$$

[9] Assuming we are dealing with continuously compounded rates or short periods so that summation is appropriate.

$$= n\alpha_j + \left(\beta_j + \frac{1}{n}\tilde{w}_{jt}\right)E\left(\tilde{R}_{Mt}\right) + \left[\beta_j + a_j\frac{1}{n}E\left(\tilde{R}_{Mt}\right)\right]\tilde{\pi}_t$$

$$+ a_j\sum_\tau \tilde{\pi}_{t\tau}^2 + \sum_\tau \tilde{w}_{jt\tau}\tilde{\pi}_{t\tau} + \tilde{u}_{jt}, \tag{39}$$

where $E\left(\tilde{R}_{Mt}\right) = nE\left(\tilde{R}_{Mt\tau}\right)$ since $E\left(\tilde{R}_{Mt\tau}\right)$ is assumed constant for all $\tau, \tilde{\pi}_t = \sum_\tau \tilde{\pi}_{t\tau}$, and $\tilde{w}_{jt} = \sum_\tau \tilde{w}_{jt\tau}$. Taking expectations of (39) we have

$$E\left(\tilde{R}_{jt}\right) = n\alpha_j + \beta_j E\left(\tilde{R}_{Mt}\right) + a_j n\sigma^2\left(\tilde{\pi}_\tau\right), \tag{40}$$

where $n\alpha_j$ is the increment in portfolio returns per n-unit time interval (i.e. the observation interval) due to the manager's stock selection ability, $\beta_j E\left(\tilde{R}_{Mt}\right)$ is the return due to the average riskiness of the portfolio and $a_j n\sigma^2\left(\tilde{\pi}_\tau\right) = a_j\sigma^2\left(\tilde{\pi}\right)$ is the return due to the manager's market forecasting abilities (all these returns being per n-unit time interval).

Note now that we cannot estimate (39) directly since $\sum_\tau \tilde{\pi}^2_{t\tau}$ is not observable if we can obtain measurements only over intervals of n periods or if we do not know n (i.e. if we do not know the length of the manager's forecasting interval). However, assuming $\tilde{\pi}_\tau$ is symmetric, if we estimate the single variable regression given by (9) the large sample estimate of α_j is

$$P\,lim\,\hat{\alpha}_j = n\alpha_j + a_j n\sigma^2\left(\tilde{\pi}_{t\tau}\right) - a_j\frac{1}{n}E\left(\tilde{R}_{Mt}\right)^2 \tag{41}$$

where as before $\sigma^2\left(\tilde{\pi}_{t\tau}\right)$ is the variance of the market returns over the manager's forecasting interval. If n is large, as it will be if the manager's forecasting period is very short relative to our measurement interval, the term $a_j 1/n\,E\left(\tilde{R}_{Mt}\right)^2$ will be negligible. Thus, the intercept in (9) will be an approximately unbiased estimate of the average *total* increment in portfolio returns per n-unit time interval due to the security selection and market forecasting abilities of the manager. Hence if we know the forecasting interval is small relative to our measurement interval we can obtain an approximately unbiased estimate of the total increment in portfolio returns due to the manager's ability but we cannot break this total down into its two components. Of course, for this case if the forecasting interval is longer than our measurement interval and we know what it is we can use the quadratic estimation procedure given by eqs. (33)-(36) as long as the ratio of the two intervals is integer.[10]

Now, for the case of optimal adjustment of market forecasts if $\tilde{\pi}_{t\tau}$ is symmetrically distributed the large sample estimate of α_j obtained from running the single variable regression (9) on the data for which the observation interval spans n forecasting periods is

$$P\,lim\,\hat{\alpha}_j = n\alpha_j + \theta_j\rho^2 n\sigma^2(\tilde{\pi}_{t\tau}) - \theta_j\frac{1}{n}E(\tilde{R}_{Mt})^2\rho^2 \tag{42}$$

And, as was the case for the naive forecast model, if n is fairly large this estimate provides us with an approximately unbiased estimate of the average total increment in portfolio returns per n-unit time interval due to the security selection and market forecasting activities of the manager.

5.6. Effects of the length of forecasting interval on profit opportunities

It is interesting to note that if the manager can shorten his market forecasting horizon (i.e., increase it) his potential profits increase enormously. Let us consider first the optimal forecast adjustment model and let θ_j^n refer to the coefficient in (19) for a

[10] Under these conditions we should ordinarily be able to adjust the measurement interval to accomplish

manager forecasting over an n-unit time interval. As we have demonstrated his expected profits, P^n, *per n-unit time interval* from these activities will be

$$P^n = \theta_j^n \rho^2 \sigma^2(\tilde{\pi}_t), \tag{43}$$

where

$$\theta_j^n = \frac{1}{2\sigma^2(\tilde{v}_t)} \frac{dV(R_j)}{dE(R_j)}. \tag{44}$$

Now, if he could reduce his forecasting interval to a single period (without changing the correlation, ρ, between his forecasts and the actual market returns) his expected profits *per unit time interval* would be $\theta_j^1 \rho^2 \sigma^2(\tilde{\pi}_{t\tau})$. Thus, his total expected profits, P^1, over *n* such periods would be

$$P^1 = \theta_j^1 \rho^2 n\sigma^2(\tilde{\pi}_{t\tau}) = \theta_j^1 \rho^2 \sigma^2(\tilde{\pi}_t) \tag{45}$$

since $\sigma^2(\tilde{\pi}_t) = n\sigma^2(\tilde{\pi}_{t\tau})$. Note, however, that

$$\begin{aligned} \theta_j^n &= \frac{1}{2\sigma^2(\tilde{v}_t)} \frac{dV(R_j)}{dE(R_j)} \\ &= n\theta_j^n \end{aligned}$$

since $\sigma^2(\tilde{v}_t) = n\sigma^2(\tilde{v}_{t\tau})$. Thus, we can rewrite (45) in terms of θ_j^n for comparison with the profits of the *n* period forecast interval as:

$$\begin{aligned} P^1 &= n\,\theta_j^n \rho^2 \sigma^2(\tilde{\pi}_t) \\ &= nP^n \end{aligned} \tag{46}$$

Thus, for a given forecasting ability a manager's expected market forecasting profits over a given time interval increase in direct proportion to the number of forecasting periods in a given time interval. Identical conclusions hold for the naive forecast model as long as

this.

the coefficient d'_1 does not change. Intuitively these results make sense since what is happening is that as the forecasting period is shortened the manager is able to profit from many more fluctuations in market prices. However, lest this seem too pat and simple let us note that it is probably extremely difficult to forecast over shorter and shorter periods with the same degree of success (i.e. constant ρ).

6. Evaluation of portfolio management under the two factor asset pricing model

6.1 The two factor asset pricing model

Black (1970) has derived the equilibrium structure of security prices in a market in which no riskless borrowing or lending opportunities exist. He has shown that the expected return on any asset *j* will be given by

$$E(\tilde{r}_j) = E(\tilde{r}_Z)(1-\beta_j) + E(\tilde{r}_M)\beta_j, \tag{47}$$

where the lower case *r*'s represent total returns, $\beta_j = cov(\tilde{r}_j, \tilde{r}_M)/\sigma^2(\tilde{r}_M)$, $E(\tilde{r}_M) =$ the expected total returns on the market portfolio and $E(\tilde{r}_Z)$ represents the expected total returns on a portfolio which has a zero covariance with $\tilde{r}_M$. Vasicek (1971) has also demonstrated that (47) holds when there exist riskless lending opportunities but no riskless borrowing opportunities.

In addition Black, Jensen and Scholes (1972) have demonstrated that the ex post returns on all securities on the New York Stock Exchange in the interval 1931-65 appear to be generated by a process given by

$$\tilde{r}_{jt} = \tilde{r}_{Zt}(1-\beta_j) + \tilde{r}_{Mt}\beta_j + \tilde{u}_{jt}, \tag{48}$$

where $cov(\tilde{r}_Z, \tilde{r}_M) = cov(\tilde{r}_Z, \tilde{u}_j) = cov(\tilde{r}_M, \tilde{u}_j) = E(\tilde{u}_j) = 0$. We now shall consider the optimal incorporation of forecasts of the zero beta portfolio, $\tilde{r}_{Zt}$, and the market portfolio, $\tilde{r}_{Mt}$, into a portfolio and then go on to consider the problems involved in measuring a manager's ability in the context of the two factor model. This expanded model appears to incorporate the inconsistencies of the observed risk-return relationship with the simple form of the model (6) which have been documented by Miller and Scholes (1972), Friend and Blume (1970) and Black, Jensen and Scholes (1972).

6.2. Optimal incorporation of forecasts of $\tilde{r}_Z$ and $\tilde{r}_M$ into a portfolio

Assume again for simplicity that the investors in the manager's portfolio all have the same preference function $U\left[E\left(\tilde{r}_j\right),V\left(\tilde{r}_j\right)\right]$, on the single period return and variance of the portfolio and that the manager engages only in trading activities designed to profit from his forecasts of next period's expected return on the market and beta factors. Given his forecasts his problem then consists of determining the fraction γ_t to invest in the market portfolio and the fraction $\left(1-\gamma_t\right)$ to invest in the zero beta portfolio.[11] The expected return and variance on his portfolio are given by

$$E\left(\tilde{R}_{jt}\right)=\gamma_t\left[E\left(\tilde{R}_M\right)+\pi^*_{Mt}\right]+\left(1-\gamma_t\right)\left[E\left(\tilde{R}_Z\right)+\pi^*_{Zt}\right]$$

$$V\left(\tilde{r}_j\right)=\gamma_t^2\sigma_j^2(\tilde{\pi}_M)+\left(1-\gamma_t\right)^2\sigma_j^2(\tilde{\pi}_Z),$$

where π^*_{Mt} and π^*_{Zt} are the manager's forecast of the 'unexpected' returns on the market and zero beta portfolios based on his information set $\phi_{j,t-1}$ at time $(t-1)$,

$$\pi^*_{Mt}=E\left[\tilde{\pi}_{Mt}\middle|\phi_{j,t-1}\right]$$

$$\pi^*_{Zt}=E\left[\tilde{\pi}_{Zt}\middle|\phi_{j,t-1}\right]$$

[11] The particular zero beta portfolio in which he is interested is, of course, the one with minimum variance.

and $\sigma_j^2(\tilde{\pi}_M)$ and $\sigma_j^2(\tilde{\pi}_Z)$ are the variances of these conditional distributions (assumed constant for all *t*)

$$\sigma_j^2(\tilde{\pi}_M) = Var\left[\tilde{\pi}_M | \phi_{j,t-1}\right]$$

$$\sigma_j^2(\tilde{\pi}_Z) = Var\left[\tilde{\pi}_Z | \phi_{j,t-1}\right]$$

The manager's problem is to choose γ_t so as to

$$\max_{\gamma_t} U\left[E(\tilde{r}_{jt}), V(\tilde{r}_{jt})\right] \tag{49}$$

Differentiation with respect to γ_t yields the optimal value

$$\begin{aligned}\gamma_t = \frac{\sigma^2(\tilde{\pi}_Z)}{\sigma^2(\tilde{\pi}_M) + \sigma^2(\tilde{\pi}_Z)} &+ \frac{1}{2\left[\sigma^2(\tilde{\pi}_M) + \sigma^2(\tilde{\pi}_Z)\right]} \frac{dV(r_j)}{dE(r_j)}\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right] \\ &+ \frac{1}{2\left[\sigma^2(\tilde{\pi}_M) + \sigma^2(\tilde{\pi}_Z)\right]} \frac{dV(r_j)}{dE(r_j)}\left[\pi^*_{Mt} - \pi^*_{Zt}\right]\end{aligned} \tag{50}$$

where $dV(r_j)/dE(r_j)$ is as defined in section 4, $E(\tilde{r}_M)$ and $E(\tilde{r}_Z)$ are the unconditional expected returns on the market and zero beta portfolios (again assumed constant through time).

Again since $\beta_M = 1$ and $\beta_Z = 0$, we see that $\beta_{jt} = \gamma_t$, and since we have assumed his anticipations regarding $\sigma_j^2(\tilde{\pi}_M)$ and $\sigma_j^2(\tilde{\pi}_Z)$ are constant through time, we can rewrite (27) as

$$\beta_{jt} = \beta_j + \theta_j\left(\pi^*_{Mt} - \pi^*_{Zt}\right) \tag{51}$$

where

$$\theta_j = \frac{1}{2\left[\sigma_j^2(\tilde{\pi}_M) + \sigma_j^2(\tilde{\pi}_Z)\right]} \frac{dV(r_j)}{dE(r_j)}$$

and

$$\beta_j = \frac{\sigma^2(\tilde{\pi}_Z)}{\sigma^2(\tilde{\pi}_M) + \sigma^2(\tilde{\pi}_Z)} + \theta_j\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]$$

can be thought of as the 'target' risk level of the portfolio. Let the manager's forecasts and the realized values of the market and beta factors be bivariate normal so that

$$\tilde{\pi}_{Mt} = d_{M0} + d_{M1}\tilde{\pi}^*_{Mt} + \tilde{v}_{jt} \tag{52a}$$

$$\tilde{\pi}_{Zt} = d_{Z0} + d_{Z1}\tilde{\pi}^*_{Zt} + \tilde{s}_{jt}, \tag{52b}$$

where $\tilde{v}_{jt}$ and $\tilde{s}_{jt}$ are normally distributed forecast errors, $\tilde{\pi}_{Zt} = \tilde{r}_{Zt} - E(\tilde{r}_Z)$ and $E(\tilde{v}_{jt}) = E\left(\tilde{\pi}^*_{Mt}\tilde{v}_{jt}\right) = E(\tilde{s}_{jt}) = E\left(\tilde{\pi}^*_{Zt}\tilde{s}_{jt}\right) = 0$. Now as in the earlier single Mt z variable model if the manager is to make optimal use of his forecast π^*_{Mt} and π^*_{Zt} he will adjust them to remove any systematic biases by forming the adjusted forecasts π'_{Mt} and π'_{Zt}:

$$\pi'_{Mt} = d_{M0} + d_{M1}\pi^*_{Mt} \tag{53a}$$

$$\pi'_{Zt} = d_{Z0} + d_{Z1}\pi^*_{Zt} \tag{53b}$$

Thus substituting these adjusted forecasts into (51) the systematic risk of the portfolio is

$$\beta_{jt} = \beta_j + \theta_j(\pi'_{Mt} - \pi'_{Zt}) \tag{54}$$

and again β_j can be thought of as the manager's 'target' risk level. We note in passing that given our assumption of bivariate normality between the forecasts and outcomes of the two factors the conditional variances $\sigma_j^2(\tilde{\pi}_M)$ and $\sigma_j^2(\tilde{\pi}_Z)$ in the definition of θ_j are respectively given by:

$$\sigma_j^2(\tilde{\pi}_M) = \sigma^2(\tilde{v}_j)$$

$$\sigma_j^2(\tilde{\pi}_Z) = \sigma^2(\tilde{s}_j).$$

If the manager simply uses his raw forecasts $\tilde{\pi}^*_{Mt}$ and $\tilde{\pi}^*_{Zt}$ we can use the relations

$$\tilde{\pi}^*_{Mt} = d'_{M0} + d'_{M1}\tilde{\pi}_{Mt} + \tilde{v}'_{jt} \tag{55a}$$

$$\tilde{\pi}^*_{Zt} = d'_{Z0} + d'_{Z1}\tilde{\pi}_{Zt} + (\tilde{s}'_{jt}) \tag{55b}$$

(where $E(\tilde{v}'_{jt}) = E(\tilde{s}'_{jt}) = E(\tilde{\pi}_{Mt}\tilde{v}'_{jt}) = E(\tilde{\pi}_{Zt}\tilde{s}'_{jt}) = 0$) to obtain the systematic risk of the portfolio for the naive forecast model as:

$$\tilde{\beta}_{jt} = \beta_j + a_{Mj}\tilde{\pi}_{Mt} + a_{Zj}\tilde{\pi}_{Zt} + \tilde{w}_{jt}, \tag{56}$$

where β_j is now defined so as to incorporate the effects of d'_{M0} and d'_{Z0}, $a_{Mj} = \theta_j d'_{M1}$, $a_{Zj} = \theta_j d'_{Z1}$, and $\tilde{w}_{jt} = \theta_j(\tilde{v}'_{jt} - \tilde{s}'_{jt})$.

6.3. Measurement of the manager's abilities to increase portfolio returns

We obtain the equation which describes the process generating the portfolio returns for the optimal forecast adjustment model by substituting from (54) into (48) and adding a constant, α_j:

$$\tilde{r}_{jt} = \alpha_j + [E(\tilde{r}_Z) + \tilde{\pi}_{Zt}][1 - \beta_j - \theta_j(\tilde{\pi}'_{Mt} - \tilde{\pi}'_{Zt})]$$
$$+ [E(\tilde{r}_M) + \tilde{\pi}_{Mt}][\beta_j + \theta_j(\tilde{\pi}'_{Mt} - \tilde{\pi}_{Zt})] + \tilde{u}_{jt}. \tag{57}$$

The expected returns on the portfolio are[12]

$$E(\tilde{r}_{jt}) = \alpha_j + (1 - \beta_j)E(\tilde{r}_Z) + \beta_j E(\tilde{r}_M) + \theta_j \rho_Z^2 \sigma^2(\tilde{\pi}_Z)$$

[12] Note that by definition that π'_M is independent of π_Z and π'_Z is independent of π_M.

$$+\theta_j \rho_M^2 \sigma^2(\tilde{\pi}_M) \tag{58}$$

where ρ_Z is the correlation between the manager's forecast, $\tilde{\pi}_Z^*$, of the unexpected return on the zero beta portfolio and the actual unexpected return, $\tilde{\pi}_Z$ and ρ_M is similarly defined as the correlation between $\tilde{\pi}_M^*$ and $\tilde{\pi}_M$, the forecast and unexpected returns on the market portfolio. As in the single variable model discussed earlier we can identify the source of each of the terms contributing to the expected returns on the manager's portfolio. The first term, α_j, is the per period expected increment in portfolio returns due to the manager's security selection activities. The second and third terms involving $E(\tilde{r}_Z)$ and $E(\tilde{r}_M)$ are the returns due to the average riskiness of the portfolio, and the last two terms are the returns due to the manager's forecasting activities. Again we see that the returns due to the managers forecasting activities are directly proportional to the variance of the two factors and, given the variances, directly proportional to the coefficients of determination between his forecasts and the outcomes of the factors.

The large sample estimate of the portfolio's systematic risk, β_j, is

$$P\,lim\,\hat{\beta}_j = \frac{cov(\tilde{r}_j, \tilde{r}_M)}{\sigma^2(\tilde{r}_M)} = \beta_j + \theta_j \left[\rho_M^2 \left[E(\tilde{r}_M) - E(\tilde{r}_Z) \right] + \frac{E(\tilde{\pi}_M^3) - E(\tilde{v}_j^3)}{\sigma^2(\tilde{\pi})} \right] \tag{59}$$

and as before we see that this estimate is upward biased if $\rho_M^2 > 0$ (since $E(\tilde{r}_M)$ and $E(\tilde{r}_Z)$ are always positive (cf. Vasicek (1971)) and the third moments of $\tilde{\pi}_M$ and $\tilde{v}_j$ will be either zero or positive). If we are dealing with continuously compounded rates or with sufficiently small differencing intervals we can probably eliminate the term involving the third moments of $\tilde{\pi}_M$ and $\tilde{v}_j$. Hence we shall in the later analysis assume the distributions to be symmetric about zero and ignore these terms.

For the naive forecasting case we can obtain the portfolio return generating process by substitution from (56) into (48) and the addition of the constant α_j:

$$\tilde{r}_{jt} = \alpha_j + \left[E(\tilde{r}_Z) + \tilde{\pi}_{Zt}\right]\left(1 - \beta_j - a_{Mj}\tilde{\pi}_{Mt} + a_{Zj}\tilde{\pi}_{Zt} - \tilde{w}_{jt}\right)$$

$$+\left[E(\tilde{r}_M) + \tilde{\pi}_{Mt}\right]\left(\beta_j + a_{Mj}\tilde{\pi}_{Mt} - a_{Zj}\tilde{\pi}_{Zt} + \tilde{w}_{jt}\right) + \tilde{u}_{jt}\cdot, \tag{60}$$

The large sample estimate of the portfolio's systematic risk is

$$P\,lim\,\hat{\beta}_j = \frac{cov(\tilde{r}_j, \tilde{r}_M)}{\sigma^2(\tilde{r}_M)} = \beta_j + a_{Mj}\left[E(\tilde{r}_M) - E(\tilde{r}_Z) + \frac{E(\tilde{\pi}_M^3)}{\sigma^2(\tilde{\pi}_M)}\right] \tag{61}$$

and the expected returns on the portfolio are

$$E(\tilde{r}_{jt}) = \alpha_j + E(\tilde{r}_Z)(1 - \beta_j) + E(\tilde{r}_M)\beta_j + a_{Zj}\sigma^2(\tilde{\pi}_Z) + a_{Mj}\sigma^2(\tilde{\pi}_M), \tag{62}$$

and the interpretation of these equations is similar to that discussed above.

If we simply run a revised version of (9)

$$\tilde{r}_{jt} = \lambda_j + \beta_j\tilde{r}_{Mt} + \tilde{e}_{jt} \tag{63}$$

in an attempt to obtain an overall estimate of the manager's contribution to portfolio returns the large sample estimate of λ_j for the optimal forecast adjustment case will be

$$P\,lim\,\lambda_j = \alpha_j + E(\tilde{r}_Z)(1 - \beta_j) + \theta_j\rho_Z^2\sigma^2(\tilde{\pi}_Z)$$

$$+\theta_j\rho_M^2\sigma^2(\tilde{\pi}_M) - \theta_j\rho_M^2\sigma^2 E(\tilde{r}_M)\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]. \tag{64}$$

As we can readily see, λ_j does not provide a direct estimate of the managers contribution to portfolio returns since it includes returns due to the beta factor, $E(\tilde{r}_Z)(1 - \beta_j)$, and the last term (similar to that in the earlier simple formulation) Will cause a negative bias if the manager can forecast the returns on the market portfolio to any extent $(\rho_M^2 > 0)$

Similarly the estimate of λ in (64) for the naive forecast model is

$$P\,lim\,\lambda_j = \alpha_j + E(\tilde{r}_Z)(1-\beta_j) + a_{Zj}\sigma^2(\tilde{\pi}_Z) + a_{Mj}\sigma^2(\tilde{\pi}_M)$$

$$-a_{Mj}E(\tilde{r}_M)\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right] \qquad (65)$$

and we have identical problems here if the manager can forecast $\tilde{\pi}_{Mt}$ to some extent.

The key to solving many of the problems we have here is to obtain an unbiased estimate of β_j. For large samples we can obtain such an unbiased estimate for the naive forecasting model (as long as our return measurement interval is identical to the manager's forecasting interval) by estimating

$$\tilde{r}_{jt} = \eta_0 = \eta_1\tilde{\pi}_{Mt} + \eta_2\tilde{\pi}_{Mt}^2 + \eta_3\tilde{\pi}_{Zt} + \eta_4\tilde{\pi}_{Zt}^2 + \tilde{u}_{jt} \qquad (66)$$

Examination of (60) indicates the coefficients will be

(a) $\hat{\eta}_0 = \alpha_j + (1-\beta_j)E(\tilde{r}_Z) + \beta_j E(\tilde{r}_M)$

(b) $\hat{\eta}_1 = \beta_j + a_{Mj}\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]$

(c) $\hat{\eta}_2 = a_{Mj}$

(d) $\hat{\eta}_3 = 1 - \beta_j - a_{Zj}\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]$

(e) $\hat{\eta}_4 = a_{Zj}$, (67)

and from this we can obtain[13] two estimates of β_j

(a) $\hat{\beta}_j = \hat{\eta}_3 + 1 - \hat{\eta}_4\left[\bar{r}_M - \bar{r}_Z\right]$

(b) $\hat{\beta}_j = \eta_1 - \eta_2\left[\bar{r}_M - \bar{r}_Z\right]$ (68)

and using either one of these estimates our estimate of α_j is given by

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[13] We obviously need estimates of the mean values of the returns on the market and zero beta portfolios. The market portfolio poses no problem and procedures for obtaining efficient estimates of the mean returns on the zero beta portfolio are given in Black et al. (1972).

$$\hat{\alpha}_j = \hat{\eta}_0 - \left(1-\beta_j\right)\bar{r}_Z - \beta_j \bar{r}_M. \qquad (69)$$

and our estimate of the increment in portfolio returns due to his forecasting ability is given by

$$\hat{\eta}_2 \hat{\sigma}^2(\tilde{\pi}_M) + \hat{\eta}_4 \hat{\sigma}^2(\tilde{\pi}_Z) \qquad (70)$$

Unfortunately, as in the simple model of section 4 we cannot obtain equivalent solutions for the case where the manager follows an optimal adjustment procedure for his forecasts π^*_M and π^*_Z without having exogenous knowledge of the parameter ρ^2_M. In the interest of brevity we omit the proof here.

6.4. Performance measurement with unknown (but small) forecasting intervals

As in the simple model discussed earlier the procedures outlined above apply only if we can measure the portfolio returns over intervals identical to the manager's forecasting interval. As before let us take the forecasting interval as unity and assume that we measure returns over a number of such intervals *n*. Then $\tilde{r}_{jt} = \sum_{\tau}^{n} \tilde{r}_{jt\tau}$ and using this we find that the estimated risk coefficient β_j for the optimal forecast adjustment model is

$$P\,lim\,\hat{\beta}_j = \beta_j + \theta_j \frac{1}{n}\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]\rho^2_M, \qquad (71)$$

and for the naive forecast model is

$$P\,lim\,\hat{\beta}_j = \beta_j + a_{Mj}\frac{1}{n}\left[E(\tilde{r}_{Mt}) - E(\tilde{r}_{Zt})\right], \qquad (72)$$

where $E(\tilde{r}_{Mt})$ and $E(\tilde{r}_{Zt})$ are the expected returns of the market and zero beta portfolios over the *n*-unit measurement time interval. (We are continuing to assume all third moments are zero.) We can see that the large sample estimate $\hat{\beta}_j \cong \beta_j$ if *n*, the number of forecasting periods in our measurement interval is large.

If we estimate (63) the intercept λ_j for the optimal forecast adjustment case is

$$P\,lim\,\hat{\lambda}_j = n\alpha_j + E(\tilde{r}_Z)\left(1-\beta_j\right) + \theta_j\rho_Z^2\sigma^2(\tilde{\pi}_Z)$$

$$+\theta_j\rho_M^2\sigma^2(\tilde{\pi}_M) - \theta_j\rho^2\frac{1}{n}E(\tilde{r}_M)\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right], \quad (73)$$

and for the naive forecast model it is

$$P\,lim\,\hat{\lambda}_j = n\alpha_j + E(\tilde{r}_{Zt})\left(1-\beta_j\right) + a_{Zj}n\sigma^2(\tilde{\pi}_{Z\tau})$$

$$+a_{Mj}n\sigma^2(\tilde{\pi}_{M\tau}) - a_{Mj}E(\tilde{r}_{Mt})\frac{1}{n}\left[E(\tilde{r}_M) - E(\tilde{r}_Z)\right]. \quad (74)$$

Thus, if *n* is large we can to a close approximation for large samples estimate the *total* increment in portfolio returns due to the manager under either forecast model by:

$$\hat{\lambda}_j - \bar{\pi}_Z\left(1-\hat{\beta}_j\right) \cong n\alpha_j + \theta_j\rho_Z^2\sigma^2(\tilde{\pi}_Z) + \theta_j\rho_M^2\sigma^2(\tilde{\pi}_M) \quad (75)$$

for the optimal forecast adjustment case, and

$$\hat{\lambda}_j - \bar{\pi}_Z\left(1-\hat{\beta}_j\right) \cong n\alpha_j + a_{Zj}\sigma^2(\tilde{\pi}_Z) + a_{Mj}\sigma^2(\tilde{\pi}_M) \quad (76)$$

for the naive forecasting model. We could also run the equivalent of (9) defining $\tilde{R}_{jt} = \tilde{r}_{jt} - \tilde{r}_{Zt}$ and $\tilde{R}_{Mt} = \tilde{r}_{Mt} - \tilde{r}_{Zt}$ and then $\hat{a}$ would provide us with an approximately unbiased estimate of the RHS of (75) or (76). Thus, again, if the forecasting interval is small relative to our return measurement interval we can obtain an estimate of the total increment in portfolio returns due to the manager's ability under either forecast model even if we don't know the forecasting interval. We cannot, however, obtain separate estimates of the returns due to his security selection and forecasting abilities trader these conditions.

7. Conclusions

In considering the evaluation of a manager's ability we have examined four major questions:

(1) How should the manager optimally incorporate his market forecasts into his policy?

(2) If the manager engages in market forecasting activities, under what conditions can we evaluate his performance in his market forecasting and security selection activities separately?

(3) Does it make any difference if he does or does not optimally adjust his forecasts?

(4) Under those conditions where we cannot separately evaluate his performance in these two dimensions can we obtain unbiased measurements of the sum of the incremental portfolio returns due to both activities?

Based on our solution for the optimal utilization of market forecasts we demonstrated that if the manager's forecasts are valueless (but he nevertheless engages in trading activities designed to capitalize on his forecasts) we can still obtain unbiased estimates of his security selection ability. We also demonstrated for the naive forecasting model that if the manager can forecast market returns and if the interval over which we measure portfolio returns is identical to the manager's forecasting interval we can evaluate the manager's performance in both dimensions separately. However, if the return measurement and forecasting intervals differ we cannot accomplish this separate evaluation. In this situation we can, however, obtain to a very close approximation an estimate of the total incremental portfolio returns due to the manager's talents if his forecasting interval is small relative to the return measurement interval. This it seems is as far as we can get in measuring performance utilizing only the time series of portfolio returns and the market and zero beta portfolio returns. If we desire any more detailed measurements it appears we shall have to have much more detailed information, such as the manager's forecasts and the portfolio composition at each point in time. We also demonstrated that these same qualitative results hold under the two factor capital asset pricing model which seems to be a solution to many of the problems of performance measurement which have been encountered in the utilization of the single factor model (cf. Friend and Blume (1970) and Black, Jensen and Scholes (1972)).

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